1 - Introduction

mercoledì 25 ottobre 2023 10:00

Parabolic dynamics

Goal: to describe the dynamics of $f : (\mathbb{C}^d, 0) \circlearrowleft$ of the form:

$f(z) = z + f^{(h)}(z) + \text{h.o.t.}$ $\quad f^{(h)} \neq 0$ homogeneous of degree $h \geq 2$

$\uparrow$ tangent to the identity (TID)

Dynamics: behavior of iterates $f^n = \underbrace{f \circ \dots \circ f}_{n \text{ times}}$.

Standard goal: normal forms.

Φ might be: biholomorphic, formal, homeo.

$\overset{\text{hol}}{\cong}$ $\quad \overset{\text{for}}{\cong}$ $\quad \overset{\text{top}}{\cong}$

conjugacy equation

$$\begin{array}{ccc} (\mathbb{C}^d, 0) & \xrightarrow{\tilde{f}} & (\mathbb{C}^d, 0) \\ \simeq \uparrow \Phi & & \simeq \uparrow \Phi \\ (\mathbb{C}^d, 0) & \xrightarrow{f} & (\mathbb{C}^d, 0) \end{array} \qquad \Phi \circ f = \tilde{f} \circ \Phi \qquad f \cong \tilde{f}$$

In our case: describe $B = \{ p \in \mathbb{C}^d, \ f^n(p) \to 0 \}$ basin of attraction

Motivations

When $d = 1$: $\quad f : \mathbb{P}^1_{\mathbb{C}} \circlearrowleft$ rational map, $\deg f \geq 2$.

One can decompose the set $\mathbb{P}^1_{\mathbb{C}}$ as $\quad \mathbb{P}^1_{\mathbb{C}} = \mathcal{F}(f) \sqcup \mathcal{J}(f)$

$\uparrow$ FATOU $\quad \uparrow$ JULIA

$\mathcal{F}(f) = \{ p \mid \{ f^n \} \text{ is locally equicontinuous at } p \}$: dynamics is regular

$\mathcal{J}(f) = \mathbb{P}^1_{\mathbb{C}} \setminus \mathcal{F}(f)$: dynamics is chaotic

If $p = f^n(p) \in \mathrm{Per}(f)$, and $\lambda = f^{n\prime}(p)$ is its multiplier, then:

- $|\lambda| < 1 \Rightarrow p \in \mathcal{F}(f)$
- $|\lambda| > 1 \Rightarrow p \in \mathcal{J}(f)$
- $|\lambda| = 1$: it depends. if $\lambda = e^{2\pi i \frac{p}{q}}$ $(\lambda^q = 1) \Rightarrow p \in \mathcal{J}(f)$ (parabolic case)
  if $\lambda = e^{2\pi i \alpha}$ $\alpha \in \mathbb{R} \setminus \mathbb{Q}$: it depends on the arithmetic properties of α.

Local dynamics in 1D

$f(z) = \lambda z\,(1 + o(1))$.

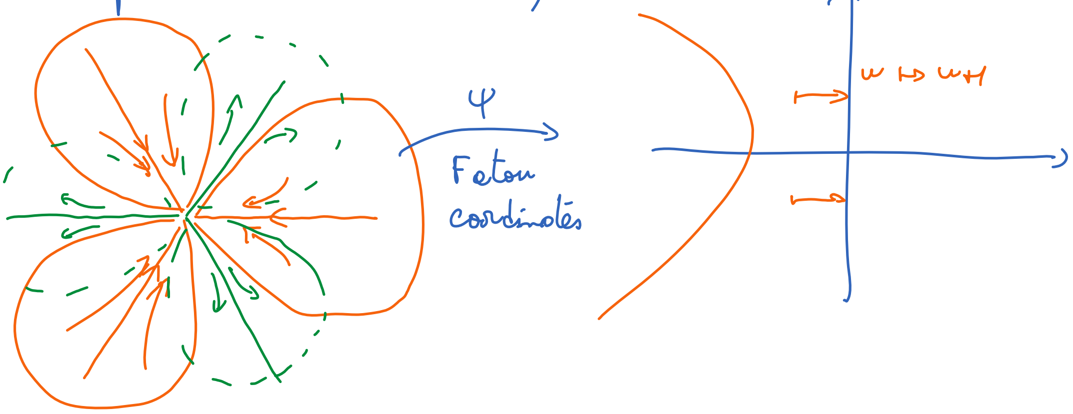
If $\lambda = 0$: $f(z) \overset{hol}{\cong} z^c \quad c \geq 2$.

If $|\lambda| \neq 1$: $f(z) \overset{hol}{\cong} \lambda z$

If $|\lambda| = 1$: $\lambda = e^{2\pi i \frac{p}{q}}$. Up to taking an iterate, we may assume $\lambda = 1$ $(q=1)$

Up to a linear change of coordinates: $f(z) = z(1 - z^r + o(z^r))$

Leau-Fatou flower theorem (~ 1930)



$h \overset{for}{\cong} z(1 - z^r + \beta z^{2r}) \quad \beta \in \mathbb{C} \quad \left(\beta = \frac{1}{2\pi i}\int_{\partial^+ D(0,\varepsilon)} \frac{dz}{z - f(z)} \text{ index}\right)$

$h \overset{top}{\cong} z(1 - z^r)$

$\cong \exp(z^r \partial_z)$

The analytic classification is more complicated to state. Depends on the horn maps: composition of Fatou coordinates for f and f^{-1} for adjacent attracting and repelling petals.

Case $\lambda = e^{2\pi i \alpha}$ $\alpha \in \mathbb{R} \setminus \mathbb{Q}$: dynamics depend on arithmetic properties of α

$\mathbb{Q}$-approximation of α	condition on approximants $\frac{p_n}{q_n}$	normal forms of F
bad	$\sum \frac{\log q_{n+1}}{q_n} < +\infty$ BRJUNO	$f \cong z \mapsto \lambda z$ $(p \in F(f))$
good	$\sum \frac{\log \log q_{n+1}}{q_n} < +\infty$ PEREZ-MARCO (and not BRJUNO)	• 2nd opt: $\exists$ small cycles $(p \in J(f))$ ("periodic" orbits close to 0)
very good	not PEREZ-MARCO	• 3rd opt: $f \not\cong z \mapsto \lambda z$ and f has no small cycles $(p \in J(f))$ ($\rightsquigarrow$ hedgehogs)

Rem: the 1D dynamics will play a role in the study of foliations, via their holonomy.

Holomorphic families and parabolic implosion

$$f_\varepsilon(z) = \lambda_\varepsilon z + z^{r+1} + z^{r+2} g_\varepsilon(z) \qquad \lambda_0 = 1, \quad \left.\frac{d\lambda_\varepsilon}{d\varepsilon}\right|_{\varepsilon=0} \neq 0.$$

f_0 has a fixed point at 0 of multiplicity $r+1$. it splits into $r+1$ simple fixed points $0, z_1, \ldots, z_r$ of multiplier

$\lambda_\varepsilon =: 1 + \mu_\varepsilon$, and $\sim 1 - r\mu_\varepsilon$

Parabolic implosion: study this phenomenon.

Applications • bifurcations (SHISHIKURA: $\dim_H(\partial \mathcal{M}) = 2$) Mandelbrot set

• polynomial mappings with $\mathrm{Leb}(J(f)) > 0$ (BUFF, CHERITAT, 2012)

• wandering domains in $d \geq 2$:

f induces a dynamics on the connected components of $F(f)$ (Fatou components). A Fatou component is wandering if its orbit is infinite.

Thm (SULLIVAN, 1985) $f:\mathbb{P}^1_{\mathbb{C}} \circlearrowleft$ rational $\deg f \geq 2$. There are no wandering Fatou components.

Thm (ASTORG, BUFF, DUJARDIN, PETERS, RAISSY, 2015) (LYUBICH)
$\exists f:\mathbb{C}^2 \to \mathbb{C}^2$ polynomial endomorphism of the form:
$$f(z,w) = \left(z + z^2 + O(z^3) + \frac{\pi^2}{4} w,\ w - w^2 + O(w^3)\right)$$
with wandering domains $\quad \nwarrow df_0 = \begin{pmatrix} 1 & \frac{\pi^2}{4} \\ 0 & 1 \end{pmatrix}$

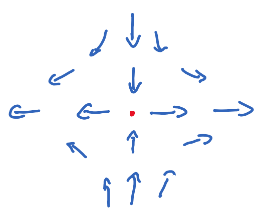
Relation with vector fields.

$$X = \Big(\underbrace{X^{(h)}}_{\not\equiv 0} + h.o.t\Big)\partial_z \quad h\geq 2$$
vector field on $(\mathbb{C}^d, 0)$ $\overset{\text{time-1 flow}}{\rightsquigarrow}$ $f = \exp X$ T.I.D of order h.

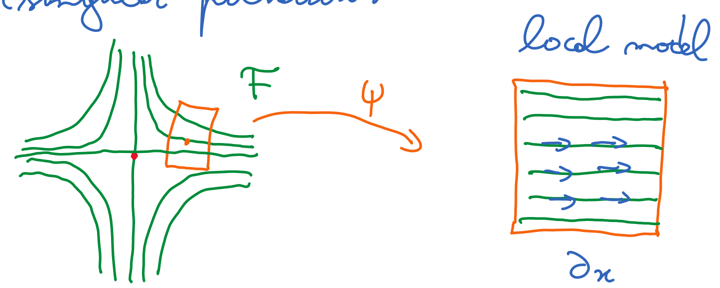
$\exists!\, X$ formal $\longleftarrow$

Vector field $X \rightsquigarrow$ Integral curves $(\gamma:(\mathbb{C},0)\to(\mathbb{C}^2,p_0),\ \gamma'(t) = X(\gamma(t)))$

$\mathcal{F}$ (singular foliation)



$\rightsquigarrow$



$$X = x\partial_x - y\partial_y \rightsquigarrow \gamma(t) = (x_0 e^t, y_0 e^{-t})$$

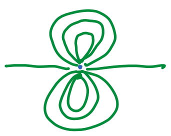
$\operatorname{Sing} X = \{p \mid X(p) = 0\} \to \operatorname{Sing} \mathcal{F}$

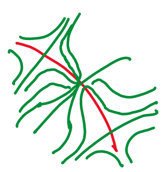
Resolution of singularities

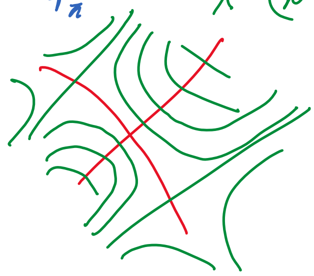
$$(\mathbb{C}^2, 0) \xleftarrow{\pi_0} X_{\pi_0} \longleftarrow X_\pi$$

$\mathcal{F}$ $\qquad \mathcal{F}_{\pi_0}$ $\qquad \mathcal{F}_\pi$

$X = (x^2 - y^2)\partial_x + 2xy\partial_y$ or similar







Thm (Seidenberg, 1968) Let $\mathcal{F}$ be a (singular) holomorphic foliation on $(\mathbb{C}^2,0)$. Then there exist a finite sequence of blow-ups of points $(\mathbb{C}^2,0) \xleftarrow{\pi_0} X_{\pi_0} \xleftarrow{\eta_{01}} X_{\pi_1} \xleftarrow{\eta_{12}} \cdots \leftarrow X_{\pi_n}$ such that:

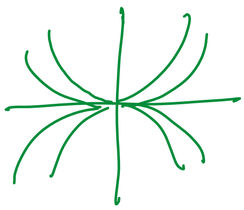
- π_j blows up a point $p_j \in \mathrm{Sing}(\mathcal{F}_{\pi_{j-1}})$
- Singularities of $\mathcal{F}_{\pi_n}$ are reduced (simple).

Reduced: $\mathcal{F}_{\pi_n}$ locally induced by χ_{π_n},
$A = \chi_{\pi_n}^{(1)}$ linear part; λ_1, λ_2 eigenvalues.
Then: A is non-nilpotent ($\lambda_1 \neq 0$), and $\lambda := \frac{\lambda_2}{\lambda_1} \in \mathbb{C} \setminus \mathbb{Q}_{>0}$.

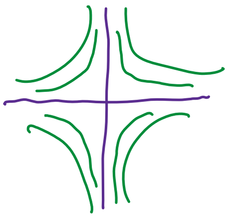
non-resonant condition

Three cases of reduced singularities:

- Poincaré: $\lambda \in \mathbb{C} \setminus \mathbb{R}_{\leq 0}$. $\rightsquigarrow$ χ is linearizable (non-resonant case)

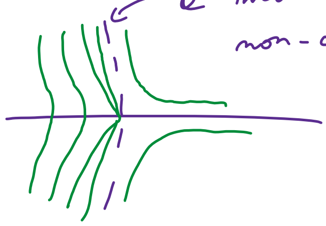


- Siegel: $\lambda \in \mathbb{R}_{<0}$ $\rightsquigarrow$ classification equivalent to germs with multiplier $e^{2\pi i\lambda}$ via holonomy



2 separatrices (= invariant analytic curve at 0)

- Saddle-node: $\lambda = 0$ $\rightarrow$



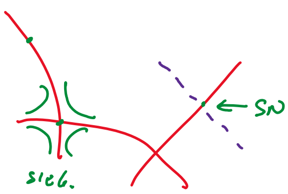
a "weak" separatrix, possibly non-convergent.

1 separatrix

Thm: (Camacho, Sad, 1982) $\forall \mathcal{F}$ foliation in $(\mathbb{C}^2,0)$, $\exists$ a separatrix.

avoid this case.

Idea: $(\mathbb{C}^2,0)$ $\cdot$ $\xleftarrow{\pi}$



Blow ups and weighted blow-ups

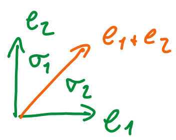
In the resolution theorem, we perform blow-ups.

$$\begin{array}{ccc} X \subset \mathbb{C}^2\times\mathbb{P}^1 & & \{(x,y),[u:v] \mid xv=yu\} \\ \downarrow \pi & & \downarrow \pi \\ (\mathbb{C}^2,0) & & (x,y) \end{array}$$

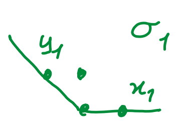
X is covered by two affine charts: $U_u=\{(x,xv),[1:v]\}$ $\quad \pi(x,v)=(x,xv)$

$U_v=\{(yu,y),[u:1]\}$ $\quad \pi(u,y)=(uy,y)$

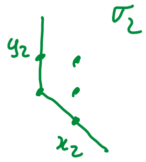
Toric construction: N



M



$$\begin{cases} x=x_1 \\ y=x_1y_1 \end{cases}$$



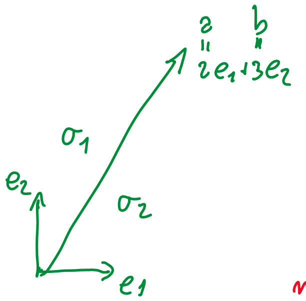
$$\begin{cases} x=x_2y_2 \\ y=y_2 \end{cases}$$

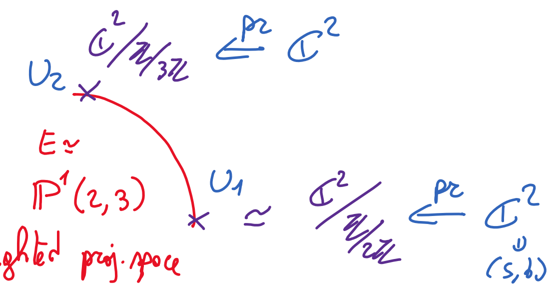
Can be also seen as the blow-up of the maximal ideal $m=\langle x,y\rangle$

This construction can be generalized to weighted blow-ups:

$(a,b)\in(\mathbb{N}_{>0})^2$

coprime





Projections: $\quad \overset{U_1}{\begin{cases} x=s^a \\ y=s^bt \end{cases}} \qquad \begin{cases} x=st^a \\ y=t^b \end{cases}$

This corresponds to the blow-up of the ideal $I=\langle x^b,y^a\rangle$

It is realized by $X_I = Cl_{Zar}\left(\{((x,y),[x^b:y^a]) : (x,y)\in\mathbb{C}^2\}\right)$

$= \{((x,y),[u:v]) : vx^b=uy^a\}$

in fact, we have two charts:

$U_u=\{((x,y),[1:v]) : vx^b=y^a\} \rightsquigarrow \quad v=t^a \quad x=s^a, \quad y=s^bt$

$U_v=\{((x,y),[u:1]) : x^b=uy^a\} \rightsquigarrow \quad u=s^b \quad y=t^b, \quad x=st^a$